

# Beals characterization of pseudodifferential operators in Wiener spaces

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## Abstract

The aim of this article is to prove a Beals type characterization theorem for pseudodifferential operators in Wiener spaces. The definition of pseudodifferential operators in Wiener spaces and a Calderón-Vaillancourt type result appear in [1]. The set of symbols considered here is the one of [1]. The Weyl calculus in infinite dimension considered here emphasizes the role of the Wick bi-symbols.

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# 1 Statement of the main result.

In quantum field theory, such as quantum electrodynamics which will be considered in a forthcoming article, the set of states of the quantized field may be chosen as a symmetrized Fock space  $\mathcal{F}_s(H_{\mathbf{C}})$  over an Hilbert space  $H$ . Among the operators acting in such spaces, those coming from the Weyl calculus in infinite dimension and recently introduced in [1] (see also in [2] the case of the large but finite dimension) may have applications to modelling the interaction of the quantized field with a fixed particle of spin 1/2. These applications will be developed in a next article, but we need some properties which are not in [1] and that we present it here.

We note by  $H$  a real separable space and by  $H_{\mathbf{C}}$  the complexified. The norm of  $H$  is noted by  $|\cdot|$  and the scalar product of two elements  $a$  and  $b$  of  $H$  is by  $a \cdot b$ . The norm of an element of  $H^2$  is denoted by  $|\cdot|$ . For all  $X = (x, \xi)$  and  $Y = (y, \eta)$  in  $H^2$ , we set

$$X \cdot \overline{Y} = (x + i\xi) \cdot (y - i\eta), \quad \sigma(X, Y) = y \cdot \xi - x \cdot \eta. \quad (1)$$

We recall that  $\mathcal{F}_s(H_{\mathbf{C}})$  is the completion of the direct sum of the subspaces  $\mathcal{F}_n$  ( $n \geq 0$ ) where  $\mathcal{F}_0$  is one dimensional and represents the vacuum, while  $\mathcal{F}_1 = H_{\mathbf{C}}$  and  $\mathcal{F}_n$  ( $n \geq 2$ ) is the  $n$ -fold symmetrized tensor product representing the  $n$  particles states. This space is not very convenient for the Weyl calculus since we have to write down integrals but it is isomorphic to some  $L^2$  space on a suitable Banach  $B$  endowed with a gaussian measure.

It is known that, for any separable real Hilbert space  $H$  there exists,

- a Banach space  $B$  containing  $H$ ,
- a gaussian measure  $\mu_{B,h}$  with variance  $h$  on the  $\sigma$ -algebra of the Borel sets of  $B$ , for all  $h > 0$ ,

satisfying some assumptions we formulate here in saying that  $(i, H, B)$  is an abstract Wiener space (where  $i$  is the injection from  $H$  into  $B$ ). See [11][12][19] and [1] for precise conditions which should be fulfilled by  $B$ . See also [13] (example 2, p. 92) for a standard way of construction of a space  $B$  satisfying the assumptions.

Identifying  $H$  with its dual, one has,

$$B' \subset H' = H \subset B. \quad (2)$$

If  $H$  is finite dimensional, we have  $B = H$  and for all Borel sets  $\Omega$  in  $H$ ,

$$\mu_{H,h}(\Omega) = (2\pi h)^{-\dim(E)/2} \int_{\Omega} e^{-\frac{|y|^2}{2h}} dy. \quad (3)$$

In the general case, the symmetrized Fock space  $\mathcal{F}_s(H_{\mathbf{C}})$  ([23],[22]) is isomorphic to the space  $L^2(B, \mu_{B,h/2})$

(see [17][24]). The complexified  $H_{\mathbf{C}} \subset \mathcal{F}_s(H_{\mathbf{C}})$  is identified with a closed subset of  $L^2(B, \mu_{B, h/2})$  which in field theory is the subspace corresponding to the states of the field with exactly one particle.

The Weyl calculus in infinite dimension of [1] allows to associate to some suitable functions  $F$  on the Hilbert space  $H^2$ , bounded and unbounded operators in  $\mathcal{F}_s(H)$  (or in  $L^2(B, \mu_{B, h/2})$ ). Let us first recall the assumptions filled by functions  $F$ .

**Definition 1.1.** *Let  $(i, H, B)$  be a Wiener space satisfying (2). We choose a Hilbert basis  $(e_j)_{(j \in \Gamma)}$  of  $H$ , each vector belonging to  $B'$ , indexed by a countable set  $\Gamma$ . Set  $u_j = (e_j, 0)$  and  $v_j = (0, e_j)$  ( $j \in \Gamma$ ). A multi-index is a map  $(\alpha, \beta)$  from  $\Gamma$  into  $\mathbb{N} \times \mathbb{N}$  such that  $\alpha_j = \beta_j = 0$  excepted for a finite number of indices. Let  $M$  be a nonnegative real number,  $m$  a nonnegative integer and  $\varepsilon = (\varepsilon_j)_{(j \in \Gamma)}$  a family of nonnegative real numbers. One denotes by  $S_m(M, \varepsilon)$  the set of bounded continuous functions  $F : H^2 \rightarrow \mathbf{C}$  satisfying the following conditions. For every multi-index  $(\alpha, \beta)$  such that  $0 \leq \alpha_j \leq m$  and  $0 \leq \beta_j \leq m$  for all  $j \in \Gamma$ , the following derivative,*

$$\partial_x^\alpha \partial_\xi^\beta F = \left[ \prod_{j \in \Gamma} \partial_{u_j}^{\alpha_j} \partial_{v_j}^{\beta_j} \right] F \quad (4)$$

*is well defined, continuous on  $H^2$  and satisfies, for every  $(x, \xi)$  in  $H^2$ ,*

$$\left| \partial_x^\alpha \partial_\xi^\beta F(x, \xi) \right| \leq M \prod_{j \in \Gamma} \varepsilon_j^{\alpha_j + \beta_j}. \quad (5)$$

For each summable sequence  $(\varepsilon_j)$ , the first step in [1] is to associate to each function  $F$  in  $S_2(M, \varepsilon)$ , a quadratic form  $Q_h^{weyl}(F)$  on a dense subset  $\mathcal{D}$  (see Definition 2.1 above), and not an operator on the above Hilbert spaces.

One may also associate a quadratic form  $Q_h^{weyl}(F)$  on  $\mathcal{D}$  with symbols  $F$  which are not in the above set, in particular if they are not bounded. To do it, it is sufficient that the two conditions below are satisfied:

(H1) The function  $F : H^2 \rightarrow \mathbf{C}$  has a stochastic extension  $\tilde{F} : B^2 \rightarrow \mathbf{C}$  in  $L^1(B^2, \mu_{B^2, h/2})$  (see definition 4.4 of [1] which recall and adapt a previous definition of L. Gross [11]).

(H2) The action on  $|\tilde{F}|$  of the following heat operator

$$(H_{h/2}|\tilde{F}|)(X) = \int_{B^2} |\tilde{F}(X + Y)| d\mu_{B^2, h/2}(Y) \quad X \in H^2 \quad (6)$$

is polynomially bounded, i.e., it satisfies for  $m \geq 0$  and  $C > 0$ ,

$$(H_{h/2}|\tilde{F}|)(X) \leq C(1 + |X|)^m \quad (7)$$

(that is to say that the norm in formula (12) in [1] is finite).

In Theorem 2.2, we recall the construction of  $Q_h^{weyl}(F)$  in a slightly simplified way, but the construction in [1] uses the analog in infinite dimension of Wigner functions which may have its own interest. The hypotheses 1 and 2 are satisfied if  $F$  belongs to  $S_2(M, \varepsilon)$ , the sequence  $(\varepsilon_j)$  being summable. Inequality (7) is then satisfied with  $C = M$  and  $m = 0$ . See others examples in Section 2.

Next, as shown in [1] (Theorem 1.4), if  $F$  belongs to  $S_2(M, \varepsilon)$  then  $Q_h^{weyl}(F)$  is the quadratic form of a bounded operator in  $L^2(B, \mu_{B, h/2})$  or equivalently, bounded in  $\mathcal{F}_s(H_{\mathbf{C}})$ . In addition, this operator satisfies, if  $0 < h < 1$ ,

$$\|Op_h^{weyl}(F)\| \leq M \prod_{j \in \Gamma} (1 + 81\pi h S_\varepsilon \varepsilon_j^2) \quad (8)$$

where

$$S_\varepsilon = \sup_{j \in \Gamma} \max(1, \varepsilon_j^2). \quad (9)$$

The hypothesis (H2) in Theorem 1.4 in [1], which not mentioned here, is always satisfied if  $F$  belongs to  $S_2(M, \varepsilon)$  and if the sequence  $(\varepsilon_j)$  is summable (Proposition 8.4 in [1]).

We have now to define and to compute, commutators of these operators with momentum and position operators. In finite dimension  $n$ , theirs compositions and commutators are classically defined as operators from  $\mathcal{S}(\mathbb{R}^n)$  into  $\mathcal{S}'(\mathbb{R}^n)$ . In our case,  $\mathcal{S}(\mathbb{R}^n)$  is replaced by space  $\mathcal{D}$  of Definition 2.1. In the absence of an analog of  $\mathcal{S}'(\mathbb{R}^n)$ , we prefer instead to use quadratic forms on  $\mathcal{D}$  (see [22]). We then consider mappings  $(f, g) \rightarrow A(f, g)$  on  $\mathcal{D} \times \mathcal{D}$  that are linear in  $f$  and antilinear in  $g$ . A notion of continuity is given in Section 2.

One may define two compositions (left and right) of a quadratic form  $Q$  on the space  $\mathcal{D}$  of Definition 2.1 with an operator  $A : \mathcal{D} \rightarrow \mathcal{D}$  whose formal adjoint  $A^*$  also maps  $\mathcal{D}$  into  $\mathcal{D}$ . One set, for all  $f$  and  $g$  in  $\mathcal{D}$ ,

$$(Q \circ A)(f, g) = Q(Af, g), \quad (A \circ Q)(f, g) = Q(f, A^*g). \quad (10)$$

One then define the commutator  $[A, Q]$  and  $(\text{ad}A)Q$  as the following quadratic form,

$$[A, Q](f, g) = Q(f, A^*g) - Q(Af, g). \quad (11)$$

Thus, one can define the iterated bracket  $(\text{ad}A_1) \dots (\text{ad}A_n)Q$  if  $A_1, \dots, A_n$  are operators from  $\mathcal{D}$  into  $\mathcal{D}$ .

We see in Proposition 2.3 that one may associate with each continuous linear form  $G$  on  $H^2$ , not only a quadratic form  $Q_h^{weyl}(G)$ , but also an operator  $Op_h^{weyl}(G)$  from  $\mathcal{D}$  to  $\mathcal{D}$ . This Weyl operator is the Segal field, up to a numerical factor, and may be directly defined in  $\mathcal{F}_s(H)$  using creation and annihilation operators, without using the Weyl calculus. In particular, when  $F(x, \xi) = a \cdot x$  with  $a$  in  $H$ , the corresponding Weyl operator will be denoted  $Q_h(a)$  (position operator). When  $F(x, \xi) = b \cdot \xi$ , when  $b$  in  $H$ , the operator will be denoted  $P_h(b)$  (momentum operator).

If  $F$  belongs to  $S_m(M, \varepsilon)$  and  $G$  is a continuous linear form on  $H^2$  then Proposition 2.6 allows us to extend the following result which is well-known in finite dimension,

$$[Q_h^{weyl}(F), Op_h^{weyl}(G)] = \frac{h}{i} Q_h^{weyl}(\{F, G\}). \quad (12)$$

In particular, if  $(e_j)$  is the Hilbertian basis of  $H$  chosen to define our sets of symbols then equality (12) gives,

$$\begin{aligned} [Q_h(e_j), Q_h^{weyl}(F)] &= -\frac{h}{i} Q_h^{weyl} \left( \frac{\partial F}{\partial \xi_j} \right), \\ [P_h(e_j), Q_h^{weyl}(F)] &= \frac{h}{i} Q_h^{weyl} \left( \frac{\partial F}{\partial x_j} \right). \end{aligned}$$

One may iterate and consider iterated commutators while restricting ourselves to some set of multi-indices. We denote by  $\mathcal{M}_m$  the set of pairs  $(\alpha, \beta)$  where  $\alpha = (\alpha_j)_{(j \in \Gamma)}$  and  $\beta = (\beta_j)_{(j \in \Gamma)}$  are sequences of nonnegative integers such that  $\alpha_j = \beta_j = 0$  except for a finite number of indices  $j$ , and such that  $\alpha_j \leq m$  and  $\beta_j \leq m$  for all  $j \in \Gamma$ . One associates to each multi-index  $(\alpha, \beta)$  the following iterated commutator,

$$(\text{ad} P_h)^\alpha (\text{ad} Q_h)^\beta Q_h^{weyl}(F) = \prod_{j \in \Gamma} (\text{ad} P_h(e_j))^{\alpha_j} \prod_{k \in \Gamma} (\text{ad} Q_h(e_k))^{\beta_k} Q_h^{weyl}(F).$$

In the same way, if  $F$  is in  $S_m(M, \varepsilon)$  and if  $(\alpha, \beta)$  is in  $\mathcal{M}_p$ ,  $p \leq m - 2$ ,

$$(\text{ad} P_h)^\alpha (\text{ad} Q_h)^\beta Q_h^{weyl}(F) = (-1)^{|\beta|} (h/i)^{|\alpha+\beta|} Q_h^{weyl}(\partial_x^\alpha \partial_\xi^\beta F).$$

From Theorem 1.4 in [1], the above Weyl quadratic form is associated to a bounded operator in  $L^2(B, \mu_{B, h/2})$ , denoted as below and verifying,

$$\|(\text{ad} P)^\alpha (\text{ad} Q)^\beta Op_h^{weyl}(F)\| \leq M \prod_{j \in \Gamma} (1 + 81\pi h S_\varepsilon \varepsilon_j^2) \prod_{j \in \Gamma} (h \varepsilon_j)^{\alpha_j + \beta_j}. \quad (13)$$

The purpose of this work is to prove the reciprocal statement, as Beals [3] did in finite dimension (see also [5][6] and [7] for adaptations to other classes of symbols in finite dimension).

**Theorem 1.2.** *Let  $(i, H, B)$  be a Wiener space satisfying (1). Let  $A_h$  be a bounded operator in  $L^2(B, \mu_{B, h/2})$ . Let  $(e_j)$  ( $j \in \Gamma$ ) a Hilbertian basis of  $H$  consisting of elements in  $B'$ . Let  $M > 0$  and let  $(\varepsilon_j)_{(j \in \Gamma)}$  a summable sequence of real numbers. Let  $m \geq 2$ . Suppose that, for all  $(\alpha, \beta)$  in  $\mathcal{M}_{m+4}$ , the commutator  $(\text{ad} P)^\alpha (\text{ad} Q)^\beta A_h$  (being a priori defined as a quadratic form on  $\mathcal{D}$ ) is bounded in  $L^2(B, \mu_{B, h/2})$  and that,*

$$\|(\text{ad} P)^\alpha (\text{ad} Q)^\beta A_h\| \leq M \prod_{j \in \Gamma} (h \varepsilon_j)^{\alpha_j + \beta_j}. \quad (14)$$

*Then, if  $0 < h < 1$ , there exists a function  $F_h$  in  $S_m(M', \varepsilon)$  with,*

$$M' = M \prod_{j \in \Gamma} (1 + K S_\varepsilon^2 h \varepsilon_j^2) \quad (15)$$

*where  $K$  is a universal constant, and  $S_\varepsilon$  is defined in (9), such that the Weyl operator  $Op_h^{weyl}(F)$  associated to  $F$  is equal to  $A_h$ .*

Section 2 introduces various results concerning the Weyl calculus in infinite dimension intended to be used in an upcoming work. Sections 3 to 7 are devoted to proof of Theorem 1.2. Section 8 applies this theorem to composition of two operators defined by the Weyl calculus. We show that the composition is also defined by this calculus, but we do not give any results on the possible asymptotic expansion of its symbol, this result being used in a forthcoming article.

## 2 Weyl calculus in infinite dimension.

### 2.1 Coherent states.

For  $X = (a, b)$  in  $H^2$ , and all  $h > 0$ , one defines  $\Psi_{X,h}$  the corresponding coherent state ([4][8][10]), they belong to  $\mathcal{F}_s(H_{\mathbf{C}})$  and are defined by,

$$\Psi_{(a,b),h} = \sum_{n \geq 0} \frac{e^{-\frac{|a|^2 + |b|^2}{4h}}}{(2h)^{n/2} \sqrt{n!}} (a + ib) \otimes \cdots \otimes (a + ib). \quad (16)$$

In view of the isomorphism from  $\mathcal{F}_s(H_{\mathbf{C}})$  in  $L^2(B, \mu_{B,h/2})$ , each element  $a$  of  $H \subset \mathcal{F}_s(H_{\mathbf{C}})$  is seen as a function in  $L^2(B, \mu_{B,h/2})$  denoted  $x \rightarrow \sqrt{h} \ell_a(x)$ . When  $a$  is in  $B' \subset H$ , one has  $\ell_a(x) = a(x)$ . When  $a$  is in  $H$ , it is approximated by a sequence  $(a_j)$  in  $B'$ , we then show that the sequence  $\ell_{a_j}$  is a Cauchy sequence in  $L^2(B, \mu_{B,h/2})$  and we denote by  $\ell_a$  its limit. With the same isomorphism, the coherent state  $\Psi_{(a,b),h}$  defined in (16) becomes,

$$\Psi_{X,h}(u) = e^{\frac{1}{h} \ell_{(a+ib)}(u) - \frac{1}{2h} |a|^2 - \frac{i}{2h} a \cdot b}, \quad X = (a, b) \in H^2, \quad \text{a.e. } u \in B. \quad (17)$$

We see, for all  $X = (x, \xi)$  and  $Y = (y, \eta)$ , with the notation (1), that

$$\langle \Psi_{Xh}, \Psi_{Yh} \rangle = e^{-\frac{1}{4h}(|X|^2 + |Y|^2) + \frac{1}{2h} X \cdot Y}. \quad (18)$$

In particular,

$$|\langle \Psi_{Xh}, \Psi_{Yh} \rangle| = e^{-\frac{1}{4h}|X-Y|^2}. \quad (19)$$

We call Segal Bargmann transform ([15]) of  $f$  the function

$$(T_h f)(X) = \frac{\langle f, \Psi_{Xh} \rangle}{\langle \Psi_{0h}, \Psi_{Xh} \rangle}, \quad X \in H^2. \quad (20)$$

We know that  $T_h f$  admits a stochastic extension  $\tilde{T}_h f$  in  $L^2(B^2, \mu_{B^2,h})$  and we know that,  $\tilde{T}_h$  is a partial isometry from  $L^2(B, \mu_{B,h/2})$  into  $L^2(B^2, \mu_{B^2,h})$ .

## 2.2 The space $\mathcal{D}$ and Wick symbols.

**Definition 2.1.** For all subspaces  $E$  of finite dimension in  $H$ ,  $\mathcal{D}_E$  denotes the space of functions  $f : B \rightarrow \mathbf{C}$  such that,

i) the function  $f$  is written under the form  $\hat{f} \circ P_E$ , where  $\hat{f}$  is a continuous function from  $E$  in  $\mathbf{C}$  and  $P_E$  is the mapping from  $B$  in  $E$  defined as follows, choosing an orthonormal basis  $\{u_1, \dots, u_n\}$  of  $E$ ,

$$P_E(x) = \sum_{j=1}^n \ell_{u_j}(x) u_j, \quad \text{a.e. } x \in B \quad (21)$$

(the map  $P_E$  is independent of the chosen basis).

ii) the function  $E^2 \ni X \rightarrow \langle f, \Psi_{Xh} \rangle$  (scalar product in  $L^2(B, \mu_{B,h/2})$ ) is in the Schwartz space  $\mathcal{S}(E^2)$ .

We shall denote by  $\mathcal{D}$  the union of all spaces  $\mathcal{D}_E$ .

We observe that the coherent states belong to  $\mathcal{D}$ . The condition ii) is equivalent to say that the function  $\hat{f}$  of i) is such that the function

$$E \ni u \rightarrow \hat{f}(u) e^{-\frac{|u|^2}{2h}} \quad (22)$$

belongs to  $\mathcal{S}(E)$ . One says that a quadratic form  $Q$  on  $\mathcal{D}$  is *continuous* if, for all  $E \subset H$  of finite dimension, there exists  $C > 0$  and  $m \geq 0$  such that, for all  $f$  and  $g$  in  $\mathcal{D}_E$ ,

$$|Q(f, g)| \leq CI(E, m)(f) I(E, m)(g) \quad (23)$$

where

$$I(E, m)(f) = \int_{E^2} |\langle f, \Psi_{Xh} \rangle| (1 + |X|)^m dX. \quad (24)$$

One says that a linear mapping  $T$  in  $\mathcal{D}$  is continuous if, for all  $E \subset H$  of finite dimension, there exists  $F \subset H$  of finite dimension such that  $f \in \mathcal{D}_E$  implies  $Tf \in \mathcal{D}_F$  and if, for all integer  $m$ , there exists  $C$  and  $m'$  such that,

$$I(F, m)(Tf) \leq CI(E, m')(f). \quad (25)$$

We shall recall the definition of the Wick symbol and bi-symbol. If  $Q$  is a quadratic form on  $\mathcal{D}$ , we denote by  $S_h(Q)$  the function defined on  $H^2$  by,

$$S_h(Q)(X, Y) = \frac{Q(\Psi_{X,h}, \Psi_{Y,h})}{\langle \Psi_{X,h}, \Psi_{Y,h} \rangle}. \quad (26)$$

If  $Q(f, g) = \langle Af, g \rangle$ , where  $A$  is an bounded operator in the Fock space  $\mathcal{F}_s(H_{\mathbf{C}})$ , or equivalently in  $L^2(B, \mu_{B,h/2})$ , then the symbol  $S_h(Q)$  will be also denoted  $S_h(A)$ . Let us recall that, if  $X = (x, \xi)$  is identified with  $x + i\xi$ , then the function  $S_h(A)$  is Gateaux holomorphic in  $X$  and antiholomorphic in  $Y$ .

We denote by  $\sigma_h^{wick}(Q)$  the restriction to the diagonal of the above function,

$$\sigma_h^{wick}(Q)(X) = Q(\Psi_{X,h}, \Psi_{X,h}). \quad (27)$$

### 2.3 Definition of the Weyl calculus in infinite dimension.

If  $H = B = \mathbb{R}^n$  and if, say,  $F$  is a  $C^\infty$  function on  $\mathbb{R}^{2n}$  bounded together with all its derivatives, one associates with  $F$  an operator  $Op_h^{weyl}(F)$  satisfying,

$$\begin{aligned} S_h(Op_h^{weyl}(F))(X, Y) &= \int_{\mathbb{R}^{2n}} F(Z) e^{\frac{i}{h}(X \cdot \bar{Z} + \bar{Y} \cdot Z - X \cdot \bar{Y})} d\mu_{h/2}(Z) \\ &= e^{\frac{i}{4h}|X-Y|^2} \int_{\mathbb{R}^{2n}} F\left(Z + \frac{X+Y}{2}\right) e^{\frac{i}{2h}((\xi-\eta) \cdot z - (x-y) \cdot \zeta)} d\mu_{\mathbb{R}^{2n}, h/2}(Z). \end{aligned} \quad (28)$$

This equality is proved in Unterberger [25] and we use it for an extension to the infinite dimensional spaces.

The first issue is that, the function  $F$  is defined on  $H^2$  according the Definition 1.1, and giving a meaning in infinite dimension to an integral such as the one in (28), we have to integrate over  $B^2$ , where  $(i, H, B)$  is a Wiener space. Indeed, in infinite dimension,  $H^2$  cannot be endowed with a gaussian measure which corresponds to its own norm.

We have to be able to extend the function  $F$ , defined on  $H^2$ , to a function  $\tilde{F}$  defined on  $B^2$ . In general it is not a density extension but a type of extension introduced by L. Gross and named *stochastic extension*. It may be found in [1] (Definition 4.4) where we recall a definition of this notion adapted to our purposes. From Proposition 8.4 of [1], we know that each function  $F$  in  $S_1(M, \varepsilon)$  admits a stochastic extension  $\tilde{F}$  in  $L^1(B^2, \mu_{B^2, h/2})$  at least if the sequence  $(\varepsilon_j)$  is summable. Moreover, the proof of Proposition 8.4 of [1] shows that any linear form  $F$  on  $H^2$  has a stochastic extension  $\tilde{F}$  in  $L^1(B^2, \mu_{B^2, h/2})$ .

By analogy with (28), one expect to associate with each function  $F$  satisfying the hypotheses (H1) and (H2) of Section 1, a quadratic form  $Q_h^{weyl}(F)$  on  $\mathcal{D}$ , with bi-symbol  $S_h(Q_h^{weyl}(F))$  of form,

$$\Phi(X, Y) = e^{\frac{i}{4h}|X-Y|^2} \int_{B^2} \tilde{F}\left(Z + \frac{X+Y}{2}\right) e^{\frac{i}{2h}(\ell_{\xi-\eta}(z) - \ell_{x-y}(\zeta))} d\mu_{B^2, h/2}(Z). \quad (29)$$

**Theorem 2.2.** *Let  $F : H^2 \rightarrow \mathbf{C}$  be a function satisfying the hypotheses (H1) and (H2) of Section 1 with  $m \geq 0$ . Let  $\tilde{F}$  be the stochastic extension of  $F$  in  $L^1(B^2, \mu_{B^2, h/2})$ . Then,*

*i) The integral (29) converges and verifies,*

$$|\Phi(X, Y)| \leq C e^{\frac{1}{4h}|X-Y|^2} \left(1 + \frac{|X+Y|}{2}\right)^m. \quad (30)$$



In addition, this function is Gateaux holomorphic in  $X$  and anti-holomorphic in  $Y$ .

ii) There is a continuous quadratic form  $Q_h^{weyl}(F)$  on  $\mathcal{D}$  such that  $S_h(Q_h^{weyl}(F)) = \Phi$ , i.e.,

$$S_h(Q_h^{weyl}(F))(X, Y) = e^{\frac{1}{4h}|X-Y|^2} \int_{B^2} \tilde{F}\left(Z + \frac{X+Y}{2}\right) e^{\frac{i}{2h}(\ell_{\xi-\eta}(z) - \ell_{x-y}(\zeta))} d\mu_{B^2, h/2}(Z). \quad (31)$$

*Proof.* i) The convergence of the integral (29) and the estimate (30) follow from hypothesis (H2). By a change of variables (cf [1][19]), the function  $\Phi$  may be also written as,

$$\Phi(X, Y) = \int_{B^2} \tilde{F}(Z) e^{\frac{1}{h}(\ell_X(\bar{Z}) + \ell_{\bar{Y}}(Z) - X \cdot \bar{Y})} d\mu_{h/2}(Z). \quad (32)$$

We deduce that it is holomorphic in  $X$  and anti-holomorphic in  $Y$ .

ii) For all  $f$  and  $g$  in  $\mathcal{D}_E$ , where  $E \subset H$  is a subspace of finite dimension, set

$$Q(f, g) = \int_{E^4} \Phi(X, Y) e^{\frac{1}{2h}X \cdot \bar{Y}} (T_h f)(X) \overline{(T_h g)(Y)} d\mu_{E^4, h}(X, Y). \quad (33)$$

Using (30) we see that, for all  $f$  and  $g$  in  $\mathcal{D}_E$ ,

$$|Q(f, g)| \leq C(2\pi h)^{-2\dim E} \int_{E^4} | \langle f, \Psi_{X, h} \rangle | | \langle g, \Psi_{Y, h} \rangle | (1 + |X|)(1 + |Y|) d\lambda(X, Y)$$

where  $\lambda$  is the Lebesgue measure. Consequently, for all  $f$  in  $\mathcal{D}_E$ , the integral defining  $Q(f, g)$  converges. When  $f$  and  $g$  belongs to  $\mathcal{D}_E$ , they also are in  $\mathcal{D}_F$ , for all subspace  $F$  containing  $E$ . If  $F$  contains  $E$ , then we denote by  $S$  the orthogonal set to  $F$  in  $E$ , and  $(X_E, X_S)$  the variable of  $F^2$ . The transform  $T_h f$  is a function on  $F^2$ , independent of the variable  $X_S$ . We remark that,

$$\int_{S^4} \Phi(X_E + X_S, Y_E + Y_S) e^{\frac{1}{2h}X_S \cdot \bar{Y}_S} d\mu_{S^4, h}(X_S, Y_S) = \Phi(X_E, Y_E).$$

Indeed, the function in the integral is holomorphic in  $X_S$ , anti-holomorphic in  $Y_S$ , and its integral is equal to its value at  $X_S = Y_S = 0$ . Consequently the definition of  $Q(f, g)$  is indeed coherent, whether that  $f$  and  $g$  are seen as functions in  $\mathcal{D}_E$  or in  $\mathcal{D}_F$ . Let us show that the bi-symbol of  $Q$  is  $\Phi$ . We have, for all  $X = (x, \xi)$  and  $Y = (y, \eta)$  in  $H^2$ , if  $E$  is the subspace spanned by  $x, \xi, y$  and  $\eta$ ,

$$\frac{Q(\Psi_{Xh}, \Psi_{Yh})}{\langle \Psi_{Xh}, \Psi_{Yh} \rangle} = \int_{E^4} \Phi(U, V) \mathcal{B}_h(X, Y, U, V) d\mu_{E^4, h}(U, V)$$

where  $\mathcal{B}_h$  is a kind of reproducing kernel,

$$\mathcal{B}_h(X, Y, U, V) = e^{\frac{1}{2h}(X \cdot \bar{U} + U \cdot \bar{V} + V \cdot \bar{Y} - X \cdot \bar{Y})}. \quad (34)$$

In a standard way, we have, if  $\Phi$  is holomorphic in  $X$ , anti-holomorphic in  $Y$ ,

$$\int_{E^4} \Phi(U, V) \mathcal{B}_h(X, Y, U, V) d\mu_{E^4, h}(U, V) = \Phi(X, Y). \quad (35)$$

It suffice to make the change of variables  $U = X + S$ ,  $V = Y + T$ , and to apply the mean formula. We then deduce that the bi-symbol of  $Q$  is indeed  $\Phi$ .

□

When  $F$  belongs to  $S_2(M, \varepsilon)$ , where the sequence  $(\varepsilon_j)$  is summable, we have proved in [1] that the quadratic form  $Q_h^{weyl}(F)$  is associated with a bounded operator.

## 2.4 Weyl symbol and Wick symbol.

It is sufficient to restrict equality (31) to the diagonal  $Y = X$  to see that,

$$\sigma_h^{wick}(Q_h^{weyl}(F))(X) = \int_{B^2} \tilde{F}(Z + X) d\mu_{B^2, h/2}(Z). \quad (36)$$

For all  $t > 0$ , the operator

$$(H_t F)(X) = \int_{B^2} \tilde{F}(X + Y) d\mu_{B^2, t}(Y) \quad (37)$$

is considered as the heat operator. In the above and below integrals on  $B^2$ ,  $\tilde{F}(X + Y)$  denotes the stochastic extension on  $B^2$  of  $H^2 \ni Y \rightarrow F(X + Y)$  for each  $X$  in  $H^2$ , which exists since it satisfies the same hypotheses as  $F$ . We then can write,

$$\sigma_h^{wick}(Q_h^{weyl}(F)) = H_{h/2} F. \quad (38)$$

Equality (38) extends the standard fact in finite dimension, that the Wick symbol is obtained from the Weyl symbol by the action of the heat operator. From Kuo [19] (Theorem 6.2) or Gross [14] (Proposition 9), the function  $H_t F$  is continuous on  $H^2$ . If  $H$  is of finite dimension, we have  $B = H$ ,  $\tilde{F} = F$ , and  $H_t F = e^{(t/2)\Delta} F$ . Note that,

$$\sup_{X \in H^2} |(H_t F)(X)| \leq \sup_{Z \in B^2} |\tilde{F}(Z)| = \sup_{X \in H^2} |F(X)|. \quad (39)$$

**Proposition 2.3.** *If  $F$  is in  $S_4(M, \varepsilon)$  with some chosen basis  $(e_j)$  and if the sequence  $(\varepsilon_j)$  is summable, then there exists  $C > 0$  such that, for all  $X$  in  $H^2$  and  $t$  in  $(0, 1)$ ,*

$$|(H_t F)(X) - F(X)| \leq Ct. \quad (40)$$

*Proof.* Let  $E_m$  be the subspace spanned by the  $e_j$  ( $j \leq m$ ). We apply (39) to the function  $F_m = F - F \circ \pi_{E_m}$ . We obtain, for all  $X$  in  $H^2$ ,

$$\int_{B^2} |(F \circ P_{E_m})(X + Y) - (\tilde{F}_t(X + Y))| d\mu_{B^2, t}(Y) \leq \|F - F \circ \pi_{E_m}\|_\infty$$

where  $\pi_{E_m} : H^2 \rightarrow E_m^2$  is the orthogonal projection and  $P_{E_m} : B^2 \rightarrow E_m^2$  is its stochastic extension, defined as in (21). If  $F$  is in  $S_1(M, \varepsilon)$ , we have,

$$\|F - F \circ \pi_{E_m}\|_\infty \leq 2M \sum_{j=p}^{\infty} \varepsilon_j. \quad (41)$$

For all  $m > 0$  and for all  $X$  in  $H^2$ , we have,

$$\int_{B^2} F(P_{E_m}(X + Y)) d\mu_{B^2, t}(Y) = \int_{E_m^2} F((\pi_{E_m} X) + Y) d\mu_{E_m^2, t}(Y).$$

According to standard results in finite dimension, we have for all  $a$  in  $E_m^2$ ,

$$\left| \int_{E_m^2} F(a + Y) d\mu_{E_m^2, t}(Y) - F(a) \right| \leq t \|\Delta_m F\|_\infty$$

where

$$\Delta_m = \sum_{j=1}^m \left( \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial \xi_j^2} \right).$$

We apply this inequality to  $a = \pi_{E_m}(X)$  using again (41). Consequently, for all  $t \in (0, 1)$  and  $m \geq 1$ ,

$$|(H_t F)(X) - F(X)| \leq 2Mt \sum_{j=1}^m \varepsilon_j^2 + 4M \sum_{m+1}^{\infty} \varepsilon_j.$$

We deduce (40) when  $m$  goes to infinity.

□

## 2.5 Operators with linear symbol. Composition.

**Proposition 2.4.** *Let  $F$  be a continuous linear form on  $H^2$ . Let  $Q_h^{weyl}(F)$  be the quadratic form on  $\mathcal{D}$  defined in Theorem 2.2. Then, there exists an operator denoted  $Op_h^{weyl}(F)$  from  $\mathcal{D}$  into itself, such that*

$$Q_h^{weyl}(F)(f, g) = \langle Op_h^{weyl}(F)f, g \rangle, \quad (f, g) \in \mathcal{D}^2. \quad (42)$$

*Proof.* Let  $f$  be in  $\mathcal{D}_E$ , where  $E \subset H$  is of finite dimension. As in Definition 2.1, we may write,  $f = \widehat{f} \circ P_E$ , where the function (22) is in  $\mathcal{S}(E)$ . Let  $a$  and  $b$  in  $H$  be such that  $F(x, \xi) = a \cdot x + b \cdot \xi$ . Let  $E_1$  be the subspace spanned by  $E$ ,  $a$  and  $b$ . Set  $f_1 : E_1 \rightarrow \mathbb{C}$  the function defined by,

$$f_1(u) = (a + ib) \cdot u \widehat{f}(\pi(u)) + \frac{h}{i} (\pi(b) \cdot \nabla \widehat{f})(\pi(u)), \quad u \in E_1$$

where  $\pi : E_1 \rightarrow E$  is the orthogonal projection. We have  $Op_h^{weyl}(F)f = f_1 \circ P_{E_1}$  and this function is in  $\mathcal{D}_{E_1}$ . Thus, if  $F$  is linear, the quadratic form  $Q_h^{weyl}(F)$  is associated with a continuous operator

$Op_h^{weyl}(F)$  from  $\mathcal{D}$  into  $\mathcal{D}$ . The set of linear functions is invariant by the operator  $H_{h/2}$ . Consequently, the Wick symbol of  $Q_h^{weyl}(F)$  is also  $F$ . We may write  $F(x, \xi) = P(X) + Q(\bar{X})$ . Then, the bi-symbol of  $Q_h^{weyl}(F)$  is  $P(X) + Q(\bar{Y})$ . We have, for all  $f$  in  $\mathcal{D}_E$ , for all  $Y \in (E_1)^2$ ,

$$\begin{aligned} \langle Op_h^{weyl}(F)f, \Psi_{Yh} \rangle &= (2\pi h)^{-n} \int_{E^2} \langle f, \Psi_{Xh} \rangle \langle Op_h^{weyl}(F)\Psi_{Xh}, \Psi_{Yh} \rangle dX \\ &= (2\pi h)^{-n} \int_{E^2} \langle f, \Psi_{Xh} \rangle [P(X) + Q(\bar{Y})] \langle \Psi_{Xh}, \Psi_{Yh} \rangle dX. \end{aligned}$$

Consequently, for all integer  $m$ ,

$$\begin{aligned} (1 + |Y|)^m |\langle Op_h^{weyl}(F)f, \Psi_{Yh} \rangle| \\ \leq C(E, E_1, h) \int_{E^2} (1 + |X|)^{m+1} |\langle f, \Psi_{Xh} \rangle| (1 + |X - Y|)^{m+1} e^{-\frac{1}{4h}|Y - X|^2} dX. \end{aligned}$$

Therefore,

$$I(E_1, m)(Op_h^{weyl}(F)f) \leq C(E, E_1, m, h)I(E, m+1)(f)$$

which proves the continuity of  $Op_h^{weyl}(F)$  in  $\mathcal{D}$ . □

Let  $A$  be a continuous quadratic form on  $\mathcal{D}$ . Let  $B : \mathcal{D} \rightarrow \mathcal{D}$  be a continuous linear mapping with a linear Wick symbol. We recall that the quadratic forms  $A \circ B$ ,  $B \circ A$  and  $[A, B]$  are defined in (10) and (11).

**Theorem 2.5.** *Let  $A_h$  be a bounded operator in  $L^2(B, \mu_{B, h/2})$ , and set  $L_h$  an operator from  $\mathcal{D}$  into  $\mathcal{D}$  with a Wick symbol being a linear form  $L(x, \xi)$  on  $H^2$ . Let  $A_h \circ B_h$  be the quadratic form on  $\mathcal{D}$  of their composition defined as in Section 1. Then, we have,*

$$\begin{aligned} \sigma_h^{wick}(A_h \circ L_h) &= \sigma_h^{wick}(A_h)\sigma_h^{wick}(L_h) + \\ &\quad \frac{h}{2} \sum_{j \in \Gamma} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial \xi_j} \right) \sigma_h^{wick}(A_h) \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial \xi_j} \right) \sigma_h^{wick}(L_h) \end{aligned}$$

*This result is valid when exchanging the roles of  $A_h$  and  $L_h$ .*

*Proof.* Set  $L(x, \xi) = a \cdot x + b \cdot \xi$  with  $a$  and  $b$  in  $H$ . Let  $X$  be in  $H^2$ . There exists an unitary operator  $W_{X, h}$  such that  $\Psi_{X, h} = W_{X, h} \Psi_{0, h}$ . We have,

$$\sigma_h^{wick}(A_h \circ L_h)(X) = \langle L_h \Psi_{X, h}, A_h^* \Psi_{X, h} \rangle = \langle f, g \rangle$$

with  $f = W_{X, h}^* L_h W_{X, h} \Psi_{0, h}$  and  $g = W_{X, h}^* A_h^* W_{X, h} \Psi_{0, h}$ . Let  $T_h f$  and  $T_h g$  be the Segal Bargmann transforms of  $f$  and  $g$  defined in (20),  $\tilde{T}_h f$  and  $\tilde{T}_h g$  being their stochastic extensions in  $L^2(B^2, \mu_{B^2, h})$ . Since  $\tilde{T}_h$  is a partial isometry from  $L^2(B, \mu_{B, h/2})$  into  $L^2(B^2, \mu_{B^2, h})$ , we have

$$\sigma_h^{wick}(A_h \circ L_h)(X) = \int_{B^2} \tilde{T}_h f(Z) \overline{\tilde{T}_h g(Z)} d\mu_{B^2, h}(Z).$$

We also have,

$$\tilde{T}_h f(Z) = L(X) + \ell_{a+ib}(z - i\zeta).$$

Since  $T_h g$  is antiholomorphic then the mean formula gives,

$$\int_{B^2} \overline{\tilde{T}_h g(Z)} d\mu_{B^2,h}(Z) = \overline{\tilde{T}_h g(0)} = \langle \Psi_{0,h}, g \rangle = \sigma_h^{wick}(A_h)(X).$$

Similarly, integrating by parts (see Theorem 6.2 of Kuo [19]), for all  $\gamma$  in the complexified of  $H$ ,

$$\int_{B^2} \ell_\gamma(z - i\zeta) \overline{\tilde{T}_h g(Z)} d\mu_{B^2,h}(Z) = h\gamma \cdot (\partial_z - i\partial_\zeta) \overline{\tilde{T}_h g(0)} = h\gamma \cdot (\partial_x - i\partial_\xi) \sigma_h^{wick}(A_h)(X).$$

The proof of Theorem then follows.

□

**Proposition 2.6.** *Let  $F$  be a function in  $S_2(M, \varepsilon)$  where the sequence  $(\varepsilon_j)$  is summable and let  $L$  be a continuous linear form on  $H^2$ . Let*

$$\Phi = FL + \frac{h}{2i} \{F, L\}, \quad \Psi = FL - \frac{h}{2i} \{F, L\}. \quad (43)$$

Then,

i) *The functions  $\Phi$  and  $\Psi$  satisfy hypotheses (H1) and (H2) in Section 1*

ii) *The corresponding Weyl forms using the Theorem 2.2 satisfy, for all  $f$  and  $g$  in  $\mathcal{D}$ ,*

$$Q_h^{weyl}(\Phi)(f, g) = \langle Op_h^{weyl}(L)f, Op_h^{weyl}(F)^*g \rangle,$$

$$Q_h^{weyl}(\Psi)(f, g) = \langle Op_h^{weyl}(F)f, Op_h^{weyl}(L)^*g \rangle.$$

*Proof.* i) Using the linearity of  $G$  and the estimates  $\int_B |\ell_a(X)| |\ell_b(X)| d\mu_{B,h/2}(X) \leq C|a||b|$ , the existence of  $L^1$  stochastic extensions are obtained similarly as in the proof of the Proposition 8.4 of [1]. The polynomial estimate on the semigroup uses that the stochastic extension of  $X \rightarrow F(X)a.X$  is  $\tilde{F}\ell_a$  with  $\int_B |\ell_a(X)| d\mu_{B,h/2}(X) \leq C|a|$ .

ii) We may write  $L(x, \xi) = a \cdot x + b \cdot \xi$  with  $a$  and  $b$  in  $H$ . From (37),

$$(H_{h/2}FL)(X) = (H_{h/2}F)(X)L(X) + \int_{B^2} \tilde{F}(X+Y)(\ell_a(y) + \ell_b(\eta)) d\mu_{B^2,h/2}(Y).$$

Integrating by parts,

$$(H_{h/2}FL)(X) = (H_{h/2}F)(X)L(X) + \frac{h}{2} \int_{B^2} \tilde{G}(X+Y) d\mu_{B^2,h/2}(Y)$$

where  $G(x, \xi) = (a \cdot \partial_x + b \cdot \partial_\xi)F$ . In other words,

$$(H_{h/2}FL)(X) = (H_{h/2}F)(X)L(X) + \frac{h}{2}(a \cdot \partial_x + b \cdot \partial_\xi)(H_{h/2}F)(X).$$

Since  $H_{h/2}$  leaves  $F$  invariant, this may be written as,

$$(H_{h/2}F)(H_{h/2}L) + \frac{h}{2} \sum_{j \in \Gamma} \left[ \frac{\partial H_{h/2}F}{\partial x_j} \frac{\partial H_{h/2}L}{\partial x_j} + \frac{\partial H_{h/2}F}{\partial \xi_j} \frac{\partial H_{h/2}L}{\partial \xi_j} \right].$$

Similarly,

$$H_{h/2}\{F, L\} = \{H_{h/2}F, L\} = \{H_{h/2}F, H_{h/2}L\}.$$

Consequently, if  $\Phi$  is defined in (43) then

$$H_{h/2}\Phi = (H_{h/2}F)(H_{h/2}L) + \frac{h}{2} \sum_{j \in \Gamma} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial \xi_j} \right) (H_{h/2}F) \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial \xi_j} \right) (H_{h/2}L).$$

From Theorem 2.5,  $H_{h/2}\Phi$  is the Wick symbol of the composition of the two operators with Wick symbols being  $H_{h/2}F$  and  $H_{h/2}L$ , that is to say,  $Op_h^{weyl}(F)$  and  $Op_h^{weyl}(G)$ . The proposition is then a consequence of the following Lemma.

□

**Lemma 2.7.** *Two continuous quadratic forms on  $\mathcal{D}$  with the same Wick symbol are equal.*

*Proof.* Let  $A$  be a continuous quadratic form on  $\mathcal{D}$  which Wick symbol vanishes identically. Let  $X$  and  $Y$  be in  $H^2$ . Set,

$$\varphi(\lambda, \mu) = S_h(A) \left( \frac{X+Y}{2} + \lambda \frac{X-Y}{2}, \frac{X+Y}{2} + \mu \frac{X-Y}{2} \right).$$

This function on  $\mathbf{C}^2$  is holomorphic in  $\lambda$ , anti-holomorphic in  $\mu$ , and identically vanishing if  $\lambda = \mu$ . It is then identically vanishing and the equality  $\varphi(1, -1) = 0$  shows that  $S_h(A)(X, Y) = 0$ . The bi-symbol of  $A$  is identically vanishing. Let  $f$  and  $g$  in  $\mathcal{D}_E$  where  $E \subset H$  is a subspace of finite dimension  $n$ . Let  $C$  and  $m$  be the constants such that we have (23) for all  $f$  and  $g$  in  $\mathcal{D}_E$ . Denote by  $D(E, m)$  of functions  $f$  such that the integral  $I(E, m)(f)$  is finite, where  $I(E, m)(f)$  is given in (24). We also have

$$f = (2\pi h)^{-n} \int_{E^2} \langle f, \Psi_{Xh} \rangle \Psi_{Xh} dX$$

and similarly for  $g$ . Then applying [26] (Section V.5) one obtains  $A(f, g)$  vanishes.

□

## 2.6 Unbounded operators. Sobolev spaces.

We denote by  $W$  the completion of  $\mathcal{D}$  for the following norm,

$$\|u\|_W^2 = \|u\|^2 + \sum_{j \in \Gamma} \|(Q_h(e_j) + iP_h(e_j))u\|^2.$$

Using annihilation operators, one has  $Q_h(e_j) + iP_h(e_j) = \sqrt{2h}a_h(e_j)$ . Using the number operator  $N = \sum a_h^*(e_j)a_h(e_j)$ , one has  $\|u\|_W^2 = \langle (I + 2hN)u, u \rangle$  (See also [18] and [20] for other Sobolev spaces in infinite dimension).

**Proposition 2.8.** *i) For all  $(a, b)$  in  $H^2$ , let  $F_{a,b}(q, p) = a \cdot q + b \cdot p$ . Then the operator  $Op_h^{weyl}(F_{ab})$  from  $\mathcal{D}$  into itself, may be extended to an operator from  $W$  in  $L^2(B, \mu_{B,h/2})$  and we have,*

$$\|Op_h^{weyl}(F_{ab})u\| \leq C(|a| + |b|) \|u\|_W. \quad (44)$$

*ii) Let  $F$  in  $S_3(M, \varepsilon)$ . Then the operator  $A_h = Op_h^{weyl}(F)$  is bounded from  $W$  into  $W$ .*

*Proof.* i) Point i) follows from estimates in Dereziński-Gérard [9], Lemma 2.1 or Lemma 2.3. The operator  $Op_h^{weyl}(F_{ab})$  is then denoted by  $\Phi_S(a + ib)$ .

ii) For all  $u$  in  $W$  and for all  $j$  in  $\Gamma$ , we have from Proposition 2.6,

$$(Q_h(e_j) + iP_h(e_j))A_h u = A_h(Q_h(e_j) + iP_h(e_j))u + hOp_h^{weyl}(G_j)u$$

with  $G_j(x, \xi) = \frac{\partial F}{\partial x_j} + i\frac{\partial F}{\partial \xi_j}$ . This function belongs to a set  $S_2(M\varepsilon_j, \varepsilon)$ . From Theorem 1.4 of [1],

$$\|A_h\| \leq M', \quad \|Op_h^{weyl}(G_j)\| \leq M'\varepsilon_j$$

where  $M'$  is independent of  $j$ . The proposition then follows. □

## 3 Reduction to finite dimension.

With a given bounded operator  $A$  in  $L^2(B, \mu_{B,h/2})$ , one always may associate a Wick symbol  $\sigma_h^{wick}(A)$ . If  $A$  verifies the hypotheses of Theorem 1.2, we shall associate a Weyl symbol  $F$  (which will depend on  $h$ ). Functions  $F$  will satisfy  $H_{h/2}F = \sigma_h^{wick}(A)$ .

We bring this study to issues related to subspaces  $E$  of finite dimension in  $B' \subset H$ . One associates two partial heat operators with each subspace  $E \subset B'$ . For any bounded continuous function  $F$  on  $H^2$  and for all  $t > 0$ , one set,

$$(H_{E,t}F)(X) = \int_{E^2} F(X + Y_E) d\mu_{E^2,t}(Y_E). \quad (45)$$

One can also define a partial heat operator acting, not on the variables of  $E^2$ , but on those of its orthogonal. The notation  $E^\perp$  now denotes,

$$E^\perp = \{x \in B, \quad u(x) = 0 \quad u \in E\}. \quad (46)$$

This heat operator related to the variables of  $(E^\perp)^2$  can only act on bounded continuous functions  $F$  on  $H^2$  with a stochastic extension  $\tilde{F}$  (bounded measurable function on  $B^2$ ). One set

$$(H_{E^\perp, t} F)(X) = \int_{(E^\perp)^2} \tilde{F}(X + Y_{E^\perp}) d\mu_{(E^\perp)^2, t}(Y_{E^\perp}). \quad (47)$$

Indeed, we know from Ramer [21] (Section 1.B), that the space  $E^\perp$  defined in (46) is also endowed with a gaussian measure. Similarly to  $H_t$ , we note that,

$$\sup_{X \in H^2} |(H_{E^\perp, t} F)(X)| \leq \sup_{X \in H^2} |F(X)|. \quad (48)$$

If  $F$  is bounded and continuous on  $H^2$  and if its stochastic extension  $\tilde{F}$  exists, then we have, from [21] (Section 1.B.),

$$H_{h/2} F = H_{E, h/2} H_{E^\perp, h/2} F. \quad (49)$$

We then consider an increasing sequence  $(\Lambda_n)$  of finite subspaces in  $\Gamma$  whose union is  $\Gamma$ . We set,

$$E(\Lambda_n) = \text{Vect}(e_j, \quad j \in \Lambda_n).$$

In Sections 4 to 7, we shall prove the following propositions.

**Proposition 3.1.** *Let  $A$  be a bounded operator in  $L^2(B, \mu_{B, h/2})$  satisfying the hypotheses of Theorem 1.2. Then,*

i) *the function  $\sigma_h^{wick}(A)$  is in the set  $S_{m+4}(M, \varepsilon)$ .*

ii) *Setting,*

$$P_{E(\Lambda_n)}(x, \xi) = \left( \sum_{j \in \Lambda_n} e_j(x) e_j, \sum_{k \in \Lambda_n} e_k(\xi) e_k \right), \quad (x, \xi) \in B^2$$

*and by denoting  $\|\cdot\|_\infty$  the supremum norm on  $H^2$ , we have,*

$$\|\sigma_h^{wick}(A) - \sigma_h^{wick}(A) \circ P_{E(\Lambda_n)}\|_\infty \leq 2M \sum_{j \notin \Lambda_n} \varepsilon_j. \quad (50)$$

**Proposition 3.2.** *Let  $A$  be a bounded operator in  $L^2(B, \mu_{B, h/2})$  satisfying the hypotheses in Theorem 1.2. Then, for all  $n$ , there exists a continuous bounded function  $F_n$  on  $H^2$  such that, if  $0 < h < 1$ ,*

i) *We have*

$$H_{E(\Lambda_n), h/2} F_n = \sigma_h^{wick}(A). \quad (51)$$



ii) The function  $F_n$  is in  $S_m(M_n, \varepsilon)$  with

$$M_n = M \prod_{j \in \Lambda_n} (1 + KS_\varepsilon^2 h \varepsilon_j^2) \quad (52)$$

where  $K$  is a numerical constant and  $S_\varepsilon$  is defined in (9).

iii) If  $n < p$  then the function  $F_n - F_p$  is in  $S_m(M_{np}, \varepsilon)$  where

$$M_{np} = M \left[ \sum_{j \in \Lambda_p \setminus \Lambda_n} K(1 + hS_\varepsilon^2)^2 h \varepsilon_j^2 \right] \prod_{j \in \Lambda_p} (1 + KS_\varepsilon^2 h \varepsilon_j^2). \quad (53)$$

These propositions will be proved in Sections 4 to 7. Let us verify that Theorem 1.2 follows from these propositions. From Proposition 3.2, the sequence  $(F_n)$  converges to a function  $F$  in  $S_m(M', \varepsilon)$  where  $M'$  is defined in (15). Let us show that  $H_{h/2}F = \sigma_h^{wick}(A)$ . From Proposition 8.4 in [1], the functions  $F_n$  have stochastic extensions  $\tilde{F}_n$ . Then, we may apply the operator  $H_{E(\Lambda_n)^\perp, h/2}$  to both sides of equality (51). We obtain from (51) and (49),

$$H_{h/2}F_n = H_{E(\Lambda_n)^\perp, h/2} \sigma_h^{wick}(A). \quad (54)$$

Let us now take the limit as  $n$  goes to infinity. We have from the point iii) of Proposition 3.2,

$$|F_n(X) - F(X)| \leq M \left[ \sum_{j \notin \Lambda_n} K h \varepsilon_j^2 \right] \prod_{j \in \Gamma} (1 + K h \varepsilon_j^2).$$

From (48) we see that, in the sense of the uniform convergence,

$$\lim_{n \rightarrow \infty} H_{h/2}F_n = H_{h/2}F. \quad (55)$$

We shall also check that,

$$\lim_{n \rightarrow \infty} H_{E(\Lambda_n)^\perp, h/2} \sigma_h^{wick}(A) = \sigma_h^{wick}(A). \quad (56)$$

Indeed, setting,  $\Psi = \sigma_h^{wick}(A)$ , we have

$$\|\Psi - H_{E(\Lambda_n)^\perp, h/2} \Psi\|_\infty \leq \|\Psi - \Psi \circ P_{E(\Lambda_n)}\|_\infty + \|H_{E(\Lambda_n)^\perp, h/2}(\Psi - \Psi \circ P_{E(\Lambda_n)})\|_\infty.$$

We have used the fact that  $H_{E(\Lambda_n)^\perp, h/2}(\Psi \circ P_{E(\Lambda_n)}) = \Psi \circ P_{E(\Lambda_n)}$ . The limit in (56) follows from (48)(50) and of point ii) in Proposition 3.1. Using (54)(55)(56) we obtain  $H_{h/2}F = \sigma_h^{wick}(A)$ . Since the function  $F$  is in  $S_m(M', K\varepsilon)$  then a Weyl quadratic form is associated with, by Theorem 2.2, and a bounded operator  $Op_h^{weyl}(F)$  associated with, by Theorem 1.4 of [1]. From (38), the Wick symbol of this operator is  $H_{h/2}F$ . Consequently the operators  $Op_h^{weyl}(F)$  and  $A$  have the same Wick symbol. From Lemma 2.7, these two operators are equal.

Once Propositions 3.1 and 3.2 proved, we have indeed found a function  $F$  in  $S_m(M', K\varepsilon)$  whose corresponding Weyl operator equals to  $A$ . Theorem 1.2 is then a consequence of Propositions 3.1 and 3.2.

□

## 4 Proof of Proposition 3.1.

Let  $A$  be a bounded operator  $A$  in  $L^2(B, \mu_{B,h/2})$  satisfying the hypotheses of Theorem 1.2. From Theorem 2.5, we have,

$$\sigma_h^{wick}([Q_h(ej), A]) = ih \frac{\partial}{\partial \xi_j} \sigma_h^{wick}(A), \quad \sigma_h^{wick}([P_h(ej), A]) = -ih \frac{\partial}{\partial \xi_j} \sigma_h^{wick}(A). \quad (57)$$

For all bounded operator  $B$ , one has,  $|\sigma_h^{wick}(B)(X)| \leq \|B\|$ . Consequently, if  $A$  verifies the hypotheses of Theorem 1.2 one deduces estimates, for each multi-index  $(\alpha, \beta)$  in  $\mathcal{M}_{m+4}$ ,

$$|\partial_x^\alpha \partial_\xi^\beta \sigma_h^{wick}(A)(x, \xi)| \leq M \prod_{j \in \Gamma} \varepsilon_j^{\alpha_j + \beta_j},$$

which prove point i) of Proposition 3.1. We deduce,

$$|\sigma_h^{wick}(A)(x, \xi) - \sigma_h^{wick}(A)(P_{E(\Lambda_n)}(x, \xi))| \leq 2M \sum_{j \notin \Lambda_n} \varepsilon_j^{\alpha_j + \beta_j},$$

which proves Proposition 3.1. We shall also need analogous estimates on the bi-symbol. One deduces from (57) these estimates by setting, for all  $j \in \Gamma$ ,

$$\frac{\partial}{\partial X_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial \xi_j} \right), \quad \frac{\partial}{\partial \bar{Y}_j} = \frac{1}{2} \left( \frac{\partial}{\partial y_j} + i \frac{\partial}{\partial \eta_j} \right).$$

With these notations, one has,

$$S_h([Q_h(e_j), A])(X, Y) = -h \left( \frac{\partial}{\partial X_j} - \frac{\partial}{\partial \bar{Y}_j} \right) (S_h A)(X, Y), \quad (58)$$

$$S_h([P_h(e_j), A])(X, Y) = -ih \left( \frac{\partial}{\partial X_j} + \frac{\partial}{\partial \bar{Y}_j} \right) (S_h A)(X, Y). \quad (59)$$

Consequently, for all multi-indices  $(\alpha, \beta)$ ,

$$S_h((\text{ad} P_h)^\alpha (\text{ad} Q_h)^\beta A)(X, Y) = c_{\alpha\beta} h^{|\alpha+\beta|} (\partial_x + \partial_y)^\alpha (\partial_\xi + \partial_\eta)^\beta S_h(A)(X, Y) \quad (60)$$

where  $|c_{\alpha\beta}| = 1$ . With (19), we deduce that

$$|(\partial_x + \partial_y)^\alpha (\partial_\xi + \partial_\eta)^\beta S_h(A)(X, Y)| \leq h^{-|\alpha+\beta|} e^{-\frac{1}{4h}|X-Y|^2} \|(\text{ad} P_h)^\alpha (\text{ad} Q_h)^\beta A\|. \quad (61)$$

## 5 Finite dimensional analysis.

We consider here the case where  $H$  is a real Hilbert space with finite dimension  $n$ . Let  $A$  be an operator satisfying hypothesis of Theorem 1.2. Let  $\Phi = S_h A$  its bi-symbol, defined in (26). We have seen that  $\Phi(X, Y)$  is holomorphic in  $X$ , anti-holomorphic in  $Y$ . From (61), the following norm is finite,

$$N_h^{(2)}(\Phi) = \sum_{(\alpha, \beta) \in \mathcal{M}_2} \|e^{-\frac{1}{4h}|X-Y|^2} (\partial_x + \partial_y)^\alpha (\partial_\xi + \partial_\eta)^\beta \Phi\|_\infty, \quad (62)$$

where  $\|\cdot\|_\infty$  is the supremum norm. Note again that a choice of particular basis has been made.

One introduces in distributions sense an integral transform giving the Weyl symbol  $F$  of  $A$  starting from the bi-symbol  $\Phi$ , and give estimates on  $F$ . This integral is not converging but has to be understood as an oscillatory integrals (see Hörmander [16]). This leads to a proof of Beals's theorem in finite dimension (see Unterberger [25]). Setting,

$$K_h^{Beals}(X, Y, Z) = e^{-\frac{1}{h}(Z-Y) \cdot (\overline{Z}-\overline{X}) - \frac{1}{2h}|X-Y|^2}. \quad (63)$$

**Theorem 5.1.** *Let  $H$  be a real Hilbert space of finite dimension  $n$ . Set  $(X, Y) \rightarrow \Phi(X, Y)$  a function on  $H^2 \times H^2$  which is holomorphic in  $X$  and anti-holomorphic in  $Y$ , such that the norm  $N_h^{(2)}(\Phi)$  defined in (62) is finite (for some orthonormal basis). Then,*

*i) The following integral transform defines, a priori in the sense of distributions, a function  $B_h\Phi$  which is bounded and continuous on  $H^2$ ,*

$$(B_h\Phi)(Z) = 2^n(2\pi h)^{-2n} \int_{H^4} \Phi(X, Y) K_h^{Beals}(X, Y, Z) dX dY. \quad (64)$$

*Moreover, this function satisfies,*

$$\|B_h\Phi\|_\infty \leq K^n N_h^{(2)}(\Phi) \quad (65)$$

*ii) Moreover, one has,*

$$(H_{h/2} B_h\Phi)(Z) = \Phi(Z, Z). \quad (66)$$

*Proof of i).* We follow the method of Unterberger [25]. The change of variables

$$X = Z + S + \frac{T}{2}, \quad Y = Z + S - \frac{T}{2}$$

allows to rewrite (64) as,

$$(B_h\Phi)(Z) = 2^n(2\pi h)^{-2n} \int_{H^4} \Psi(S, T, Z) K_h(S, T) dS dT \quad (67)$$

with

$$\Psi(S, T, Z) = \Phi\left(Z + S + \frac{T}{2}, Z + S - \frac{T}{2}\right) \quad (68)$$

$$K_h(S, T) = e^{-\frac{1}{h}|S|^2 - \frac{i}{h}\sigma(S, T) - \frac{1}{4h}|T|^2}. \quad (69)$$

Set  $S_j = (s_j, \sigma_j)$ ,  $T_j = (t_j, \tau_j)$ . Let  $L_j$  and  $M_j$  be the operators defined, for each function  $G(S, T)$ , by

$$L_j G = \left(1 + \frac{\tau_j^2}{h}\right)^{-1} e^{-\frac{1}{h}s_j^2} \left(1 - h \frac{\partial^2}{\partial s_j^2}\right) e^{\frac{1}{h}s_j^2} G$$

$$M_j G = \left(1 + \frac{t_j^2}{h}\right)^{-1} e^{-\frac{1}{h}\sigma_j^2} \left(1 - h \frac{\partial^2}{\partial \sigma_j^2}\right) e^{\frac{1}{h}\sigma_j^2} G.$$

One verifies that,

$$L_j K_h = K_h, \quad M_j K_h = K_h \quad j \leq n$$

where the function  $K_h$  defined in (69). Consequently,

$$(B_h \Phi)(Z) = 2^n (2\pi h)^{-2n} \int_{H^4} K_h(S, T) \left[ \prod_{j \leq n} {}^t L_j {}^t M_j \right] \Psi(S, T, Z) dS dT.$$

We see that,

$${}^t L_j = \left(1 + \frac{\tau_j^2}{h}\right)^{-1} \left[ a_0(s_j/\sqrt{h}) + h^{1/2} a_1(s_j/\sqrt{h}) \partial_{s_j} + h a_2(s_j/\sqrt{h}) \partial_{s_j}^2 \right]$$

with

$$a_0(s) = 3 - 4s^2, \quad a_1(s) = 4s, \quad a_2(s) = -1.$$

Similarly,

$${}^t M_j = \left(1 + \frac{t_j^2}{h}\right)^{-1} \left[ a_0(\sigma_j/\sqrt{h}) + h^{1/2} a_1(\sigma_j/\sqrt{h}) \partial_{\sigma_j} + h a_2(\sigma_j/\sqrt{h}) \partial_{\sigma_j}^2 \right]$$

Consequently,

$$\begin{aligned} |(B_h \Phi)(Z)| &\leq \sum_{(\alpha, \beta) \in \mathcal{M}_2} h^{|\alpha+\beta|/2} F_{\alpha\beta}(Z) \\ F_{\alpha\beta}(Z) &= 2^n (2\pi h)^{-2n} \int_{H^4} e^{-\frac{1}{h}|S|^2} \prod_{j \leq n} \left(1 + \frac{t_j^2}{h}\right)^{-1} \left(1 + \frac{\tau_j^2}{h}\right)^{-1} \left| a^\alpha(s/\sqrt{h}) a^\beta(\sigma/\sqrt{h}) \right| \\ &\quad |e^{-\frac{1}{4h}|T|^2} \partial_s^\alpha \partial_\sigma^\beta \Psi(S, T, Z)| dS dT \end{aligned}$$

where we have set

$$a^\alpha(s) = \prod_{j \leq n} a_{\alpha_j}(s_j).$$

There exists  $K > 0$  such that,

$$\pi^{-1/2} \int_{\mathbb{R}} e^{-s^2} |a_j(s)| ds \leq K, \quad 0 \leq j \leq 2$$

and also

$$(2\pi)^{-1/2} \int_{\mathbb{R}} (1+x^2)^{-1} dx \leq K.$$

Consequently,

$$|(B_h \Phi)(Z)| \leq K^n \sum_{(\alpha, \beta) \in \mathcal{M}_2} h^{|\alpha+\beta|/2} \sup_{(S, T) \in H^4} \left| e^{-\frac{1}{4h}|T|^2} \partial_s^\alpha \partial_\sigma^\beta \Psi(S, T, Z) \right|.$$

From the definition of  $\Psi$  in (68),

$$|(B_h \Phi)(Z)| \leq K^n \sum_{(\alpha, \beta) \in \mathcal{M}_2} h^{|\alpha + \beta|/2} \sup_{(X, Y) \in H^4} \left| e^{-\frac{1}{4h}|X-Y|^2} (\partial_x + \partial_y)^\alpha (\partial_\xi + \partial_\eta)^\beta \Phi(XY) \right|.$$

We then deduce (66) with another constant  $K$ .

*Proof of ii).* If a function  $\Psi$  on  $H^2$  is written as

$$\Psi(Z) = e^{-\frac{1}{h}|Z|^2 + \frac{1}{h}(A \cdot Z + B \cdot \bar{Z})}$$

where  $A$  and  $B$  are in  $H^2$ , and  $A \cdot Z$  denotes the bi- $\mathbf{C}$ -linear scalar product, then the action of the heat operator on  $\Psi$  verifies,

$$(H_{h/2} \Psi)(Z) = \left( e^{\frac{h}{4} \Delta} \Psi \right)(Z) = 2^{-n} e^{\frac{1}{2h}|Z|^2 + \frac{1}{2h}(A \cdot Z + B \cdot \bar{Z})} \frac{1}{2h} A \cdot B.$$

Thus,

$$(H_{h/2} K_h^{Beals}(X, Y, \cdot))(Z) = 2^{-n} \mathcal{B}_h(Z, Z, X, Y) e^{-\frac{1}{2h}(|X|^2 + |Y|^2)}$$

where  $\mathcal{B}_h$  is our type of reproducing kernel introduced in (34). Consequently, as in (35)

$$(H_{h/2} B_h \Phi)(Z) = \int_{H^4} \Phi(X, Y) \mathcal{B}_h(Z, Z, X, Y) d\mu_{H^4, h}(X, Y) = \Phi(Z, Z).$$

## 6 Proof of Proposition 3.2: first step.

For all operators  $A$  satisfying the hypotheses of Theorem 1.2 and for some subsets  $E$  of finite dimension in  $B' \subset H$ , we shall find a bounded continuous function  $\tau_{E, h}(A)$  on  $H^2$  such that

$$H_{E, h/2} \tau_{E, h}(A) = \sigma_h^{wick}(A). \quad (70)$$

It is point i) of Proposition 3.2. Moreover, we shall give estimations on this function. For all finite subsets  $I$  in  $\Gamma$ , let  $E(I)$  be the subspace of  $B' \subset H$  spanned by the  $e_j$ ,  $j \in I$ . Recall that the elements  $e_j$  ( $j \in \Gamma$ ) of our Hilbertian basis are in  $B'$ . Let  $\mathcal{M}_2(I)$  be the set of all multi-indices  $(\alpha, \beta)$  such that  $\alpha_j = \beta_j = 0$  if  $j \notin I$ , and  $\alpha_j \leq 2$  and  $\beta_j \leq 2$  if  $j \in I$ .

**Proposition 6.1.** *Let  $A$  be an operator satisfying the hypotheses in Theorem 1.2. Set  $I$  a finite subspace of  $\Gamma$ . Then, there exists a bounded continuous function  $\tau_{E(I), h}(A)$  on  $H^2$  satisfying (70). Moreover,*

$$\|\tau_{E(I), h}(A)\|_\infty \leq K^{|I|} \sum_{(\alpha, \beta) \in \mathcal{M}_2(I)} h^{-|\alpha + \beta|/2} \|(\text{ad} P_h)^\alpha (\text{ad} Q_h)^\beta A\| \quad (71)$$

where  $K$  is a numerical constant.

*Proof.* We denote  $E = E(I)$ ,  $E^\perp$  the orthogonal complement of  $E$  in  $H$ , and  $Z = (Z_E, Z_{E^\perp})$  the variable in  $H^2$ . For all  $Z_{E^\perp}$  in  $(E^\perp)^2$ , we shall apply Proposition 5.1 replacing  $H$  by  $E$ , with the following function  $\Phi$  defined on  $E^2$ ,

$$\Phi_{Z_{E^\perp}}(X_E, Y_E) = (S_h A)(X_E, Z_{E^\perp}, Y_E, Z_{E^\perp}).$$

Using again notation (64), which a priori only makes sense as an oscillatory integral on  $E^2$ , one set for all  $Z = (Z_E, Z_{E^\perp})$  in  $H^2$ ,

$$\tau_{E(I),h}(A)(Z) = 2^{\dim(E)} (2\pi h)^{-2\dim(E)} \int_{E^4} (S_h A)(X_E, Z_{E^\perp}, Y_E, Z_{E^\perp}) K_h^{Beals}(X_E, Y_E, Z_E) dX_E dY_E$$

where  $K_h^{Beals}$  is defined in (63). One may apply Theorem 5.1, choosing as an orthonormal basis of  $E = E(I)$ , the one constituted with the  $e_j$   $j \in I$ . With this choice, we have from (61),

$$N_h^{(2)}(\Phi_{Z_{E^\perp}}) \leq \sum_{(\alpha, \beta) \in \mathcal{M}_2(I)} h^{-|\alpha+\beta|/2} \|(\text{ad} P_h)^\alpha (\text{ad} Q_h)^\beta A\|$$

and the term in the right hand side is finite under hypothesis of Theorem 1.2. From Theorem 5.1, the function  $\tau_{E(I),h}(A)$  is well-defined, continuous and bounded on  $H^2$  and satisfies (70) and (71). □

## 7 Proof of Proposition 3.2: second step.

For all finite subsets  $I$  of  $\Gamma$ , let us set

$$T_{I,h} = \prod_{j \in I} (I - H_{D_j, h/2}) \quad (72)$$

where  $D_j$  is spanned by the vector  $e_j$  of our Hilbertian basis of  $H$ , and  $H_{D_j, h/2}$  is the operator defined in (45), with  $E$  replaced by  $D_j$ , thus with an integral on  $D_j^2$ . When  $I = \emptyset$ , we set  $T_{I,h} = Id$ . We denote by  $E(I)$  the subspace of  $B'$  spanned by the  $e_j$ ,  $j \in I$ . Recall that the elements  $e_j$  ( $j \in \Gamma$ ) of our Hilbertian basis of  $H$  are in  $B'$ . If  $I = \emptyset$  then set  $E(I) = \{0\}$ . For any operator  $A$  satisfying the hypotheses in Theorem 1.2 and for all subspaces  $E \subset B' \subset H$  of finite dimension, set  $\tau_{E(I),h}(A)$  the function on  $H^2$  defined in the Proposition 6.1. In particular, we may have  $E = E(I)$  with  $I$  being a finite subset of  $\Gamma$ . We choose an increasing sequence  $(\Lambda_n)$  of finite subsets of  $\Gamma$  with its union equals to  $\Gamma$ . For all  $n$ , one defines a function  $F_n$  on  $H^2$  by,

$$F_n = \sum_{I \subset \Lambda_n} T_{I,h} \tau_{E(I),h}(A). \quad (73)$$

The above sum is running over all the subsets  $I$  of  $\Lambda_n$  including the empty set. We shall show that this sequence of functions has indeed the properties announced Proposition 3.2.

Point i) One has, for all subsets  $I \subset \Lambda_n$ ,

$$H_{E(\Lambda_n),h/2} = H_{E(I),h/2} H_{E(\Lambda_n \setminus I),h/2}$$

and these operators commutes with each other and with  $T_{I,h}$ . Consequently,

$$H_{E(\Lambda_n),h/2} F_n = \sum_{I \subset \Lambda_n} T_{I,h} H_{E(\Lambda_n \setminus I),h/2} H_{E(I),h/2} \tau_{E(I),h}(A).$$

From equality (70) applied to set  $E(I)$ , one has,

$$H_{E(\Lambda_n),h/2} F_n = \sum_{I \subset \Lambda_n} T_{I,h} H_{E(\Lambda_n \setminus I),h/2} \sigma_h^{wick}(A).$$

The following equality is a variant of the binomial formula,

$$\sum_{I \subset \Lambda_n} T_{I,h} H_{E(\Lambda_n \setminus I),h/2} = Id.$$

So, we have proved equality (51), point i) of the Proposition 3.2.

Points ii) and iii) will both be a direct consequence of the following inequality. If  $A$  satisfies hypothesis in Theorem 1.2, for all  $(\alpha, \beta)$  in  $M_m$ , for any finite subset  $I$  in  $\Gamma$  and for all  $h$  in  $(0, 1)$ ,

$$\|\partial_z^\alpha \partial_\zeta^\beta T_{I,h} \tau_{E(I),h}(A)\|_\infty \leq M(K S_\varepsilon^2)^{|I|} \prod_{j \in I} h \varepsilon_j^2 \prod_{j \in \Gamma} \varepsilon_j^{\alpha_j + \beta_j} \quad (74)$$

where  $K$  is a numerical constant and  $S_\varepsilon$  is defined in (9).

It remains to prove (74). If  $H_{D_j,h/2}$  is defined in (45), with  $E$  replaced by  $D_j = \text{Vect}(e_j)$ , we may write,

$$I - H_{D_j,h/2} = \frac{h}{4} V_j (\partial_{z_j}^2 + \partial_{\zeta_j}^2)$$

where the operators  $V_j$  are bounded in the space  $C_b$  of continuous bounded functions on  $H^2$ , and are commuting with partial derivatives operators. Moreover,

$$\|V_j\|_{\mathcal{L}(C_b)} \leq 1.$$

Therefore, one may rewrite the operator  $T_{I,h}$  defined in (72) under the following form,

$$T_{I,h} = \prod_{j \in I} (h/4) V_j (\partial_{z_j}^2 + \partial_{\zeta_j}^2).$$

Let  $\mathcal{N}(I)$  be the set of multi-indices  $(\alpha, \beta)$  such that  $\alpha_j = \beta_j = 0$  if  $j \notin I$ , and if  $j \in I$ , either we have  $\alpha_j = 2$  and  $\beta_j = 0$ , or  $\alpha_j = 0$  and  $\beta_j = 2$ . Consequently,

$$\|\partial_z^\alpha \partial_\zeta^\beta T_{I,h} \tau_{E(I),h}(A)\|_\infty \leq (h/4)^{|I|} \sum_{(\gamma, \delta) \in \mathcal{N}(I)} \|\partial_z^{\alpha+\gamma} \partial_\zeta^{\beta+\delta} \tau_{E(I),h}(A)\|_\infty.$$

On verifies that,

$$\left[ \frac{\partial}{\partial x_j} + \frac{\partial}{\partial y_j} + \frac{\partial}{\partial z_j} \right] K_h^{Beals}(X, Y, Z) = 0, \quad \left[ \frac{\partial}{\partial \xi_j} + \frac{\partial}{\partial \eta_j} + \frac{\partial}{\partial \zeta_j} \right] K_h^{Beals}(X, Y, Z) = 0.$$

Consequently,

$$\partial_z^\alpha \partial_\zeta^\beta \tau_{E(I),h}(A) = \tau_{E(I),h} A_{\alpha\beta}$$

where  $A_{\alpha\beta}$  is such that,

$$(S_h A_{\alpha\beta})(X, Y) = (\partial_x + \partial_y)^\alpha (\partial_\xi + \partial_\eta)^\beta (S_h A)(X, Y).$$

From (60),

$$A_{\alpha\beta} = c_{\alpha\beta} h^{-|\alpha+\beta|} (\text{ad} P_h)^\alpha (\text{ad} Q_h)^\beta A$$

where  $|c_{\alpha\beta}| = 1$ . Then,

$$\|\partial_z^\alpha \partial_\zeta^\beta T_{I,h} \tau_{E(I),h}(A)\|_\infty \leq (h/4)^{|I|} \sum_{(\gamma,\delta) \in \mathcal{N}(I)} h^{-|\alpha+\beta+\gamma+\delta|} \|\tau_{E(I),h}((\text{ad} P_h)^{\alpha+\gamma} (\text{ad} Q_h)^{\beta+\delta} A)\|_\infty.$$

From the Proposition 6.1,

$$\begin{aligned} \|\partial_z^\alpha \partial_\zeta^\beta T_{I,h} \tau_{E(I),h}(A)\|_\infty &\leq (Kh/4)^{|I|} \sum_{(\gamma,\delta) \in \mathcal{N}(I)} \sum_{(\lambda,\mu) \in \mathcal{M}_2(I)} h^{-|\alpha+\beta+\gamma+\delta|-|\lambda+\mu|/2} \\ &\quad \|((\text{ad} P_h)^{\alpha+\gamma+\lambda} (\text{ad} Q_h)^{\beta+\delta+\mu} A)\|. \end{aligned}$$

If  $(\alpha, \beta) \in \mathcal{M}_m$ ,  $(\gamma, \delta) \in \mathcal{N}(I)$  and  $(\lambda, \mu) \in \mathcal{M}_2(I)$ , then the sum  $(\alpha + \gamma + \lambda, \beta + \delta + \mu)$  belongs to  $\mathcal{M}_{m+4}$ . From assumptions of Theorem 1.2,

$$\|\partial_z^\alpha \partial_\zeta^\beta T_{I,h} \tau_{E(I),h}(A)\|_\infty \leq M(Kh/4)^{|I|} \sum_{(\gamma,\delta) \in \mathcal{N}(I)} \sum_{(\lambda,\mu) \in \mathcal{M}_2(I)} h^{|\lambda+\mu|/2} \prod_{j \in \Gamma} \varepsilon_j^{\alpha_j + \beta_j + \gamma_j + \delta_j + \lambda_j + \mu_j}.$$

The number of multi-indices in  $\mathcal{N}(I)$  is  $2^{|I|}$ , and the number of multi-indices in  $\mathcal{M}_2(I)$  is  $9^{|I|}$ . For all multi-indices  $(\gamma, \delta) \in \mathcal{N}(I)$ , we have  $\gamma_j + \delta_j = 2$  if  $j \in I$ . If  $0 < h < 1$ , for all multi-indices  $(\lambda, \mu) \in \mathcal{M}_2(I)$ , we have  $(\sqrt{h} \varepsilon_j)^{\lambda_j + \mu_j} \leq S_\varepsilon^2$ , where  $S_\varepsilon$  is defined in (9). Consequently, we have indeed proved (74) with another universal constant  $K$ . From (73), we deduce the points ii) and iii) of the Proposition 3.2, which complete the proof of Theorem 1.2. □

## 8 Composition of operators.

**Theorem 8.1.** *Let  $F$  in  $S_{m+6}(M, \varepsilon)$  and  $G$  in  $S_{m+6}(M', \varepsilon)$  ( $m \geq 0$ ). Then there exists a function  $H_h$  in  $S_m(M'', (m+4)\varepsilon)$  such that,*

$$Op_h^{weyl}(F) \circ Op_h^{weyl}(G) = Op_h^{weyl}(H_h). \quad (75)$$



We have set,

$$(8.2) \quad M'' = MM' \prod_{j \in \Gamma} (1 + K(m+4)^2 S_\varepsilon^2 h \varepsilon_j^2)^3$$

where  $K$  is a universal constant and  $S_\varepsilon$  is defined in (9).

*Proof.* For any multi-index  $(\alpha, \beta)$  in  $\mathcal{M}_{m+4}$  we have,

$$\begin{aligned} & (\text{ad} P_h)^\alpha (\text{ad} Q_h)^\beta \left( Op_h^{\text{weyl}}(F) \circ Op_h^{\text{weyl}}(G) \right) = \\ & \sum_{\substack{\alpha' + \alpha'' = \alpha \\ \beta' + \beta'' = \beta}} \left( (\text{ad} P_h)^{\alpha'} (\text{ad} Q_h)^{\beta'} Op_h^{\text{weyl}}(F) \right) \circ \left( (\text{ad} P_h)^{\alpha''} (\text{ad} Q_h)^{\beta''} Op_h^{\text{weyl}}(G) \right). \end{aligned}$$

From (13) (with  $m$  replaced by  $m+6$ ) and similarly for  $G$ , we have, for each multi-index  $(\alpha, \beta)$  in  $\mathcal{M}_{m+4}$ ,

$$(\text{ad} P_h)^\alpha (\text{ad} Q_h)^\beta \left( Op_h^{\text{weyl}}(F) \circ Op_h^{\text{weyl}}(G) \right) \leq MM' N(\alpha, \beta) \prod_{j \in \Gamma} (1 + 81\pi h S_\varepsilon^2 \varepsilon_j^2)^2 \prod_{j \in \Gamma} (h \varepsilon_j)^{\alpha_j + \beta_j}$$

where  $N(\alpha, \beta)$  is the number of decompositions of  $(\alpha, \beta)$  as a sum of two multi-indices  $(\alpha', \beta')$  and  $(\alpha'', \beta'')$ . If  $(\alpha, \beta)$  is in  $\mathcal{M}_{m+4}$  then this number equals is smaller than  $(m+4)^{|\alpha+\beta|}$ . Consequently,  $Op_h^{\text{weyl}}(F) \circ Op_h^{\text{weyl}}(G)$  satisfies a condition similar to (14) with  $\varepsilon_j$  replaced by  $(m+4)\varepsilon_j$ . So our Theorem 8.1 is a consequence of Theorem 1.2.

□

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